

Charged shells and elementary particles

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Abstract

We review the General Relativistic model of a (quasi) point-like particle represented by a massive shell of electrically charged matter, which displays an ADM mass M equal to the electric charge $|Q|$ in the small-volume limit. We employ the Israel-Darboux's junction equations to explicitly derive this result, and then study the modifications introduced by the existence of a minimum length scale λ . For λ of the order of the Planck length (or larger), we find that the ADM mass becomes equal to the bare mass m_0 of the shell, like it occurs for the neutral case.

1 Introduction and general perspective

Elementary particles are usually viewed as being point-like in classical physics, although the stress-energy tensor of the electromagnetic and Newtonian gravitational fields then diverge at the particle's location. This divergence can be removed in General Relativity, by replacing localised sources with shells of matter [1], whose “size” and total energy, the Arnowitt-Deser-Misner (ADM) mass M , remain finite in the “point-like” limit. This result is further interpreted as the fact that General Relativity does not allow to store finite amounts of energy in a vanishingly small volume.

In Quantum Mechanics, the Heisenberg Uncertainty Principle prevents complete localisation in the phase space of Minkowskian theories. Moving forward to a semiclassical scenario, with quantum matter evolving on a classical background space-time, rigorous results and plausibility arguments suggest the emergence of a fundamental length scale, say λ [2, 3, 4, 5]. Such a scale is usually predicted to be of the order of the Planck length $\ell_p \simeq 10^{-33}$ cm, corresponding to a mass $m_p \simeq 10^{16}$ TeV, well beyond the realm of earth-based experiments. From a theoretical point of view, it is still interesting to explore the conceptual implications of the existence of a length $\lambda \sim \ell_p$ on the fundamental nature of elementary particles. More specific questions then are, for example, if black holes differ significantly from standard model particles and whether quantum transitions may occur between black holes and regular particles at the Planck scale. The latter issue is particularly relevant for the understanding of the end-point of Hawking evaporation, but might also be relevant for studying the formation of horizons inside collapsing matter.

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We shall first re-derive the results of Ref. [1] by making use of Israel-Darboux's junction equations [6], showing that (spinless) electrically charged particles with bare mass m_0 can be described in General Relativity by extremal configurations of the Reissner-Nordstroem metric with (geometrical) charge $|Q| = M$. We shall then introduce a minimum length $\lambda \gtrsim \ell_p$, thus extending the analysis previously performed for the neutral case [7], and find that the main result of Ref. [7], namely that $M \simeq m_0$, still holds for $Q \neq 0$. Some connections with recent models of the internal degrees of freedom of black holes will also be mentioned.

For simplicity, we shall mostly use units with $G = c = 1$, so that all variables have dimension of length to a given power, unless differently specified.

2 ADM shell model

Following Ref. [1], we consider the space-time generated by an infinitely thin shell of bare mass m_0 , electric charge Q and (isotropic coordinate ¹) radius $r = \epsilon$. Space will therefore be divided into an interior region ($0 \leq r < \epsilon$) and exterior region ($r > \epsilon$), relative to the shell radius $r = \epsilon$.

2.1 Interior geometry

For the interior region ($0 \leq r < \epsilon$), we shall assume flat Minkowski space-time, which, in the usual spherical coordinates with areal radius \bar{r} , of course, reads

$$ds_o^2 = -f_i dt_i^2 + f_i^{-1} d\bar{r}_i^2 + \bar{r}_i^2 d\Omega^2, \quad (2.1)$$

with $f_i = 1$.

2.2 Exterior geometry

We assume the exterior region ($r > \epsilon$) is described by the Reissner-Nordström metric, which is written, in Schwarzschild-like coordinates, as

$$ds_o^2 = -f_o dt^2 + f_o^{-1} d\bar{r}^2 + \bar{r}^2 d\Omega^2, \quad (2.2)$$

where

$$f_o = 1 - \frac{2M}{\bar{r}} + \frac{Q^2}{\bar{r}^2}, \quad (2.3)$$

and the constant M is the ADM mass of the system, as we shall show below. Let us recall the above metric displays (up to) two horizons $\bar{r} = \bar{R}_\pm$, namely

$$\bar{R}_\pm = M \pm \sqrt{M^2 - Q^2}, \quad (2.4)$$

provided the charge is small enough, that is $|Q| \leq M$. For $|Q| > M$, the metric (2.2) instead represents a naked singularity located at $\bar{r} = 0$ (if that region of space is accessible). In our case, since the outer geometry ends at $\bar{r} = \bar{r}(\epsilon) > 0$, where the shell is located, the singularity is not part of the physical space-time. Nonetheless, in the following we shall pay particular attention to those to cases, namely $|Q| \leq M$ and $|Q| > M$.

¹The precise definition of r will be given below in Section 2.2.

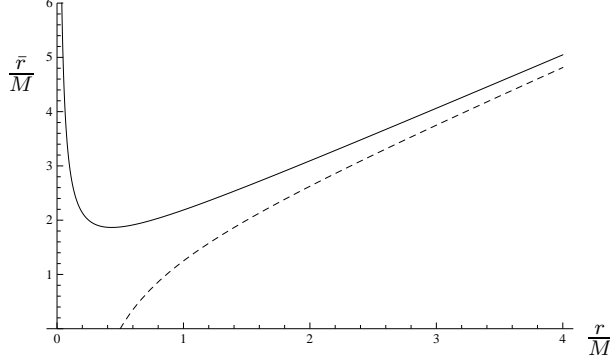


Figure 1: Radial coordinate \bar{r} as function of isotropic radius r for fixed M and $|Q| = M/2 < M$ (solid line): the throat is at $r_+/M = \sqrt{3}/4 \simeq 0.43$ with areal radius $\bar{R}_+/M = (2 + \sqrt{3})/2 \simeq 1.86$. Same function for M and $|Q| = 2M > M$ (dashed line): there is no throat and $\bar{r}(r_{\min}) = 0$ for $r_{\min}/M = 1/2$.

Before we proceed, we provide the isotropic form of the metric, namely

$$ds_o^2 = - \left(\frac{4r^2 - M^2 + Q^2}{4r^2 + 4Mr + M^2 - Q^2} \right)^2 dt^2 + \left(1 + \frac{M - Q}{2r} \right)^2 \left(1 + \frac{M + Q}{2r} \right)^2 (dr^2 + r^2 d\Omega^2) , \quad (2.5)$$

which is obtained from Eq. (2.2) with the change of radial coordinate

$$\bar{r}(r) = r \left(1 + \frac{M}{2r} \right)^2 - \frac{Q^2}{4r} = r + M + \frac{M^2 - Q^2}{4r} . \quad (2.6)$$

From this form, assuming $|Q| < M$, we see that the metric (2.5) can be used to represent a wormhole, asymptotically flat both for $r \rightarrow \infty$ and $r \rightarrow 0$. In fact, if $|Q| < M$, the areal radius \bar{r} diverges both for $r \rightarrow \infty$ and $r \rightarrow 0$, and has a minimum for

$$r_+ = \frac{1}{2} \sqrt{M^2 - Q^2} , \quad (2.7)$$

corresponding to $\bar{r}(r_+) = \bar{R}_+$ (see solid line in Fig. 1). In this case, the shell can be placed at any $r = \epsilon \geq 0$. For $|Q| > M$, there is no throat (\bar{R}_+ becomes imaginary) and the areal radius instead vanishes for finite $r = r_{\min}$ (see dashed line in Fig. 1), $\bar{r}(r_{\min}) = 0$, where

$$r_{\min} = \frac{1}{2} (|Q| - M) > 0 . \quad (2.8)$$

The metric (2.5) now does not represent a worm-hole and the shell can only have an isotropic radius $\epsilon \geq r_{\min}$. Obviously, this complication does not arise with a neutral shell, which can always be associated to a worm-hole [7]. Finally, for $|Q| = M$, the Reissner-Nordstroem metric (2.2) becomes extremal, with $\bar{R}_- = \bar{R}_+$, and Eq. (2.5) reduces to

$$ds_o^2 = - \left(1 + \frac{M}{r} \right)^{-2} dt^2 + \left(1 + \frac{M}{r} \right)^2 (dr^2 + r^2 d\Omega^2) , \quad (2.9)$$

with $\bar{r}(r) = r + M$ and $r \geq r_{\min} = -M < 0$. Note that the latter minimum value of r cannot be smoothly reached by taking $|Q| \rightarrow M$, with $|Q| > M$, in Eq. (2.8). Extremal configurations will appear to play a decisive role in Section 3.

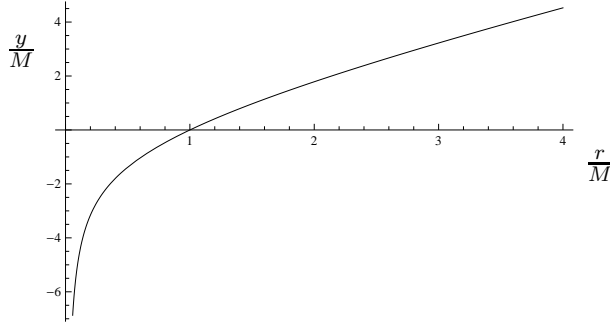


Figure 2: Gaussian normal coordinate y as function of isotropic radius r for fixed M , $|Q| = M/2$ and $y = 0$ for $r = M$.

Another useful parameterization is given by Gaussian normal coordinates, in which the spatial part of the metric looks flat along the direction perpendicular to surfaces of constant r or $\bar{r} = \bar{r}(r)$ (the usual two-spheres), that is $g_{yy} = 1$. For the metric (2.2), the coordinate y is simply defined by the equation

$$dy = \frac{d\bar{r}}{\sqrt{f_o(\bar{r})}} = \left(\frac{d\bar{r}}{dr} \right) \frac{dr}{\sqrt{f_o(\bar{r}(r))}} , \quad (2.10)$$

which, on using Eq. (2.6), yields

$$y(r) = y_0 + r + M \ln(r) + \frac{Q^2 - M^2}{4r} , \quad (2.11)$$

where y_0 is an integration constant that can be so chosen that, for example, $y = 0$ for $r = \epsilon$ (see Fig. 2). It is now easy to compute the components K_{ij} of the extrinsic curvature of a two-sphere of isotropic radius $r = x$. In particular, we shall just need ²

$$K_{\theta\theta} = \frac{1}{2} \frac{d\bar{r}^2}{dy} \Big|_{r=x} = \bar{r}(x) \sqrt{f_o(\bar{r}(x))} , \quad (2.12)$$

from which we also obtain the trace $K = K^i_i$,

$$K = 2 g^{\theta\theta} K_{\theta\theta} = 2 \frac{\sqrt{f_o(\bar{r})}}{\bar{r}} . \quad (2.13)$$

Note that, for $Q = 0$, we recover the extrinsic curvature for the neutral case, and the flat case result, if we further set $M = 0$ ³,

$$K_{\theta\theta}^0 = \bar{r} . \quad (2.14)$$

The total energy of this spherically symmetric space-time with asymptotically flat metric (for $\bar{r} \sim r \rightarrow \infty$) is given by the surface integral

$$E = - \lim_{R \rightarrow \infty} \left[\int \frac{d\theta d\phi}{16\pi} \sqrt{g^{(2)}} (K - K^0)_{\bar{r}=R} \right] , \quad (2.15)$$

²The constant y_0 in Eq. (2.11) does not affect the following result and need not be specified.

³The curvature is expressed in terms of \bar{r} , because the junction conditions will be employed below in the reference frame of the areal coordinate, and not of the isotropic r .

where $g^{(2)}$ and K are, respectively, the determinant of the two-metric and the trace of the extrinsic curvature of a two-sphere of areal radius R . The trace K^0 , obtained by embedding the two-sphere in a three-dimensional Euclidean space, yields the Minkowski “reference” energy. It is easy to see that, from Eqs. (2.2), (2.13) and (2.14), one obtains

$$E = M , \quad (2.16)$$

which shows that M is indeed the ADM mass of the system, as expected [1], and regardless of the value of Q .

2.3 Shell energy-momentum tensor

The shell matter at isotropic $r = \epsilon$ is represented by a $\delta^{(3)}$ -function energy density,

$$\sqrt{g^{(3)}} T^t_t = -\frac{m_0}{2} \sqrt{\eta^{(3)}} \delta^{(3)}(r) , \quad (2.17)$$

where $g^{(3)}$ is the determinant of the spatial metric, $\eta^{(3)} = r^4 \sin^2 \theta$, and

$$4\pi \int_0^\infty \delta^{(3)}(r) r^2 dr = 1 , \quad (2.18)$$

with $\delta^{(3)}(r) = 0$, for $|r - \epsilon| > 0$. We just recall here that $\epsilon \geq 0$ for $|Q| \leq M$, whereas $\epsilon \geq r_{\min}$ given in Eq. (2.8) for $|Q| > M$.

One can more easily describe the shell in comoving coordinates such that the (2+1)-dimensional metric on the shell reads

$$ds_\epsilon^2 = -d\tau^2 + R^2 (d\theta^2 + \sin^2 \theta d\phi^2) , \quad (2.19)$$

and the shell energy momentum tensor is diagonal,

$$T_{ij} = \text{diag} [-m_0, -R^2 p, -R^2 (\sin^2 \theta) p] , \quad (2.20)$$

where p is the surface tension.

2.4 Junction equations

We shall enforce the junction equations in the radial areal coordinate frame in which the inner, outer and shell metrics are respectively given by Eqs. (2.1), (2.2) and (2.19). Continuity of the metric across the shell, $ds_1^2 = ds_0^2 = ds_\epsilon^2$ at $r = \epsilon$, then implies

$$R = \bar{r}(\epsilon) = \bar{r}_i(\epsilon) . \quad (2.21)$$

Moreover, the discontinuity of the extrinsic curvature at the shell surface,

$$K_{\theta\theta}(r = \epsilon) - K_{\theta\theta}^0(r = \epsilon) = -m_0 , \quad (2.22)$$

leads to

$$M^2 + 2\epsilon M - 2\epsilon m_0 - Q^2 = 0 , \quad (2.23)$$

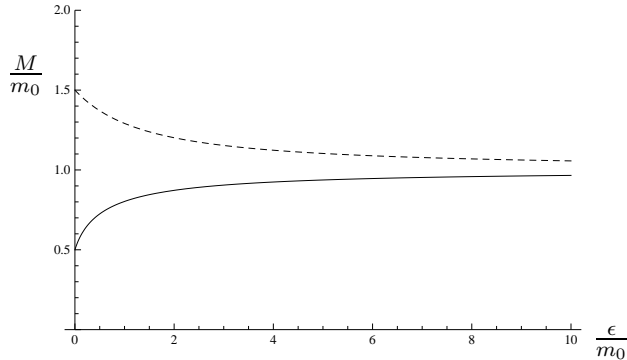


Figure 3: ADM mass M/m_0 versus the areal coordinate $r = \epsilon/m_0$ of the shell for fixed m_0 and $|Q| = m_0/2 < m_0$ (solid line) and $|Q| = 3m_0/2 > m_0$ (dashed line).

from which we obtain the final result

$$M = -\epsilon + \sqrt{\epsilon^2 + 2\epsilon m_0 + Q^2} , \quad (2.24)$$

namely an ADM mass that depends on the shell (isotropic) radius $r = \epsilon$. Note that Eq. (2.24) reduces to the analogous expression for $Q = 0$ in Refs. [1, 7], and has the expected asymptotic behaviour for large $\epsilon \gg m_0$, namely

$$M \simeq m_0 . \quad (2.25)$$

For intermediate radii, the ADM mass interpolates monotonically between m_0 and $|Q|$ (see Fig. 3).

3 Small-volume limit

In Section 3.1, we shall analyse what happens to the system when the shell isotropic radius is taken to its mathematically possible lowest value [1]. We will consider the cases $|Q| \leq M$ and $|Q| > M$ separately, so that the minimum value of ϵ is either 0 or r_{\min} . Subsequently, in Section 3.2, we shall impose a minimum length to the shell areal radius and derive physical consequences like in Ref. [7].

3.1 Classical model

For $|Q| \leq M$, one can consider a vanishing isotropic radius, $\epsilon \rightarrow 0$, and Eq. (2.24) simply yields the result of Ref. [1], namely

$$M = |Q| , \quad (3.1)$$

which shows that the bare mass m_0 does not have any gravitational effects (at large distance). In fact, one has $M = 0$ in the neutral case (see also Ref. [7] for a detailed analysis).

A vanishing isotropic radius however does not necessarily mean the shell has reduced to a point. Due to the dependence of M on ϵ given in Eq. (2.24), the shell areal radius [given by Eq. (2.6) for $r = \epsilon$] becomes

$$\bar{r}(\epsilon) = \frac{1}{2} \left(m_0 + \epsilon + \sqrt{\epsilon^2 + 2\epsilon m_0 + Q^2} \right) , \quad (3.2)$$

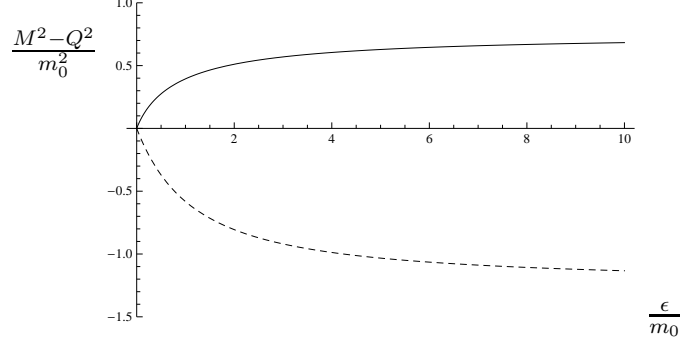


Figure 4: Difference $M^2 - Q^2$ versus ϵ for fixed m_0 and $|Q| = m_0/2 < m_0$ (solid line) and $|Q| = 3m_0/2 > m_0$ (dashed line). In the former case, the outer geometry contains a throat of radius (3.4), whereas in the latter it does not.

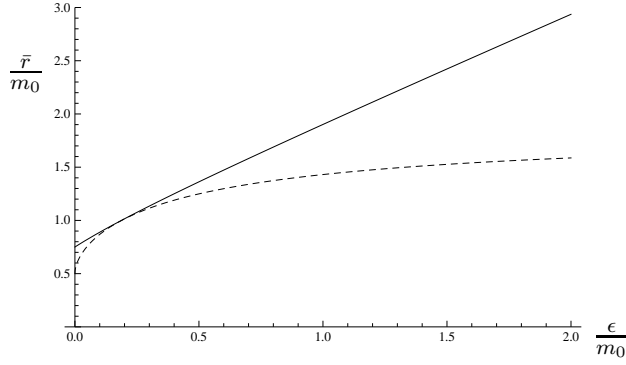


Figure 5: Areal coordinate $\bar{r} = \bar{r}(\epsilon)$ of the shell (solid line) and throat radius $\bar{R}_+(\epsilon)$ (dashed line) for fixed m_0 and $|Q| = m_0/2$.

and remains finite for vanishing isotropic shell radius $r = \epsilon \rightarrow 0$ (see Fig. 5),

$$\bar{r}(\epsilon \rightarrow 0) = \frac{m_0 + |Q|}{2} \equiv \bar{r}_0, \quad (3.3)$$

This means that the considered configuration is not really point-like in its naive sense, but has a “size” (area) determined by both charge and bare mass. Note also that the expression for the throat radius \bar{R}_+ from Eq. (2.4) becomes

$$\bar{R}_+ = -\epsilon + \sqrt{\epsilon^2 + 2\epsilon m_0 + Q^2} + \sqrt{2\epsilon} \sqrt{m_0 + \epsilon - \sqrt{\epsilon^2 + 2\epsilon m_0 + Q^2}}, \quad (3.4)$$

which is real only provided $m_0 \geq |Q|$, as can be seen from the difference $M^2(\epsilon) - Q^2$ plotted in Fig. 4. Therefore, for $m_0 > |Q|$, we find the very important consistency check that

$$\bar{r}_0 > \bar{R}_+. \quad (3.5)$$

It is actually easy to see that $\bar{r}(\epsilon) \geq \bar{R}_+(\epsilon)$ for all $\epsilon \geq 0$ (see Fig. 5), which means that the shell never disappears behind its horizon (recall that the space-time for $\bar{r} < \bar{r}_0$ is actually flat Minkowski), and this model does not describe (charged) particles as black holes, but as worm-holes. More precisely,

since

$$\lim_{\epsilon \rightarrow 0} M^2(\epsilon) = Q^2 , \quad (3.6)$$

the outer geometry becomes that of an extremal Reissner-Nordstrom space-time.

For $|Q| > M$, we can only take $\epsilon \rightarrow r_{\min}$ [see Eq. (2.8)]. From Eq. (2.24) and Fig. 4, we see that this case is generated by $|Q| > m_0$. However, for the same reason, we also see that $r_{\min} = r_{\min}(M)$ becomes

$$r_{\min} = \frac{1}{2} \left(\epsilon + |Q| - \sqrt{\epsilon^2 + 2\epsilon m_0 + Q^2} \right) , \quad (3.7)$$

and vanishes for $\epsilon \rightarrow 0$. This means that it is legitimate to take $\epsilon \rightarrow 0$ also for $|Q| > m_0$ and, since Eq. (3.1) still holds, the outer geometry of the final configuration is an extremal Reissner-Nordström metric like for $|Q| \leq m_0$.

The overall conclusion is therefore that, from the point of view of an observer placed in the region outside of the charged shell, at $\bar{r} > \epsilon$, after the limit $\epsilon \rightarrow 0$, the geometry is given by the Reissner-Nordström metric with $M = |Q|$, regardless of whether $|Q| \leq m_0$ or $|Q| > m_0$. Finally, for a neutral particle, the known limit is recovered by simply setting $Q = 0$, which yields $\bar{r}_0 = m_0/2$ and $\bar{R}_+ = 0$ [7]. This known result, if applied to elementary particles, naturally raises some questions about the implementation of the equivalence principle, since the bare shell mass m_0 becomes unobservable in this limiting process. However, elementary particles should also be described by quantum mechanics, which brings us to the next step.

3.2 Quantum models

In order to further clarify the above classical results, let us display the Newton constant $G = \ell_p/m_p$ and Planck constant $\hbar = \ell_p m_p$ explicitly. This means, for example, that the bare (geometrical) mass

$$m_0 = \ell_p \frac{\mu_0}{m_p} , \quad (3.8)$$

where μ_0 has units of mass, and the geometrical charge

$$|Q| \simeq 10^8 \ell_p \left(\frac{\mu_0}{m_p} \right) \left| \frac{q}{e} \right| . \quad (3.9)$$

For astrophysical objects, the total charge q is negligible and the condition $|Q| \ll m_0$ therefore usually holds. However, for standard model particles with an electric charge q equal to (a fraction of) the electron charge e , the geometrical charge $|Q| \simeq 10^8 m_0$, and the throat radius (3.4) becomes imaginary. Moreover, since $\mu_0 \ll m_p$, one has

$$m_0 \ll |Q| \ll \ell_p . \quad (3.10)$$

We have already shown that the classical limit $\epsilon \rightarrow 0$ can also be performed for $|Q| > m_0$, so that the left inequality is no issue. However, dealing with classical lengths below the Planck size is definitely more questionable.

Let us then repeat the analysis of the neutral case from Ref. [7], and introduce a minimum length scale λ , such that both the shell areal radius and the throat radius (when properly defined) cannot be shorter than λ ,

$$\bar{r}(\epsilon) \geq \bar{r}(\epsilon_\lambda) \equiv \bar{r}_\lambda \simeq \lambda \quad (3.11)$$

and

$$\bar{R}_+(\epsilon) \geq \bar{R}_+(\epsilon_\lambda) \equiv \bar{R}_\lambda \simeq \lambda . \quad (3.12)$$

From Eq. (3.2), we obtain

$$\epsilon_\lambda = \lambda \left(1 - \frac{m_0}{2\lambda}\right)^2 - \frac{Q^2}{4\lambda} = \lambda - m_0 + \frac{m_0^2 - Q^2}{4\lambda} , \quad (3.13)$$

and the ADM mass becomes

$$M_\lambda = m_0 \left(1 - \frac{m_0}{2\lambda}\right) + \frac{Q^2}{2\lambda} = m_0 + \frac{Q^2 - m_0^2}{2\lambda} , \quad (3.14)$$

which shows that the asymptotic expression (2.25) is recovered provided $\lambda \gg |Q|, m_0$. The limit $\epsilon \rightarrow 0$ and the classical result (3.1) can be recovered by simply setting $\lambda = \bar{r}_0$. Moreover, the analogous expressions for the neutral case in Ref. [7] are recovered by setting $Q = 0$.

Of course, since Eq. (3.5) holds in general, these results are meaningful only provided $\lambda \gtrsim \bar{r}_0$ in Eq. (3.2), that is

$$\lambda \gtrsim \begin{cases} |Q| \simeq 10^8 m_0 \sim 10^8 \ell_p \frac{\mu_0}{m_p} & \text{for } q \simeq e \\ \frac{m_0}{2} \sim \ell_p \frac{\mu_0}{m_p} & \text{for } q = 0 , \end{cases} \quad (3.15)$$

Note that both bounds above are much shorter than the Planck length for standard model particles, whose mass $\mu_0 \lesssim 10^{-16} m_p$.

In a quantum theory, we could therefore assume that

- 1) $\lambda \simeq \ell_p$, and λ is a truly fundamental length, or
- 2) $\lambda = \ell_p m_p / \mu_0 \equiv \lambda_\mu$ is the Compton length of the particle, with $\lambda_\mu \gg \ell_p$.

In either case,

$$m_0 \ll |Q| \ll \lambda , \quad (3.16)$$

and we can therefore approximate

$$M_\lambda \simeq m_0 . \quad (3.17)$$

Note this result holds irrespectively of the sign of the difference $|Q| - m_0$, as long as Eq. (3.16) is valid, and coincides with what was obtained for the neutral case in Ref. [7]. It therefore the effect of the existence of a minimum (quantum mechanical) length swaps the roles of Q and m_0 : the later becomes the observable ADM mass, in complete agreement with the literal interpretation of the equivalence principle, whereas the former remains just the source of the electric field.

4 Conclusions and outlook

We have reworked the old shell model of elementary particles of Ref. [1] and shown, in a rather pedagogical manner, why quantum corrections may be responsible for restoring the classical equivalence principle for massive charged particles, precisely in the same way it was previously derived for neutral particles in Ref. [7].

It would be tempting at this point to speculate whether there exists a different implementation of the limiting procedure $\epsilon \rightarrow 0$ and obtain, instead of a regular geometry, a true black hole space-time. Of course, one such way is the actual collapse of a homogenous sphere of dust that ends with a Schwarzschild black hole (the Oppenheimer-Snyder model [8]). Beside mathematical differences, at the quantum level, one could then devise a means to investigate transitions between the two different outcomes of the “collapse”: regular particles on one side, and black holes on the other. This would provide a basic building block for describing both the formation of black holes from the collision of quantum particles [9], and the late stages of the inverse process of Hawking evaporation [10]. One could also try to bridge the classical and quantum descriptions of such “point-like states”, by employing a specific model of black holes of the form recently proposed in Refs. [11, 12].

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